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Heisenberg evolution in a quantum theory of noncommutative fields

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Abstract: A quantum theory of noncommutative fields was recently proposed by Carmona, Cortez, Gamboa and Mendez ([1]). The implications of the noncommutativity of the fields, intended as the requirements $[\phi, \phi^{+}] = \theta\delta^{3}(x - x'), [\pi, \pi^{+}] = B\delta^{3}(x - x')$, were analyzed on the basis of an analogy with previous results on the so-called "noncommutative harmonic oscillator construction". Some departures from Lorentz symmetry turned out to play a key role in the emerging framework. We first consider the same hamiltonian proposed in [1], and we show that the theory can be analyzed straightforwardly within the framework of Heisenberg evolution equation without any need of making reference to the "noncommutative harmonic oscillator construction". We then consider a rather general class of alternative hamiltonians, and we observe that violations of Lorentz invariance are inevitably encountered. These violations must therefore be viewed as intrinsically associated with the proposed type of noncommutativity of fields, rather than as a consequence of a specific choice of hamiltonian.

Keywords: Space-Time Symmetries, Non-Commutative Geometry, Beyond Standard Model.

Contents

1. Introduction

Recently a "quantum theory of noncommutative fields" has been proposed in ref. [1], as a possible field theory generalization of noncommutative quantum mechanics [2]–[4]. The type of theory of noncommutative fields introduced in ref. [1] is of course rather different from ordinary relativistic quantum field theory, and, as we will stress, it is also rather different from a quantum field theory in noncommutative spacetime. In fact, the theory of ref. [1] is formulated in classical (commutative) spacetime, but admits nonzero equal-time commutation relations not only between the fields and their conjugated momenta, but also between the field ϕ and its adjoint ϕ^+, and between the momentum π and its adjoint π^+.

One of the primary reasons of interest in the model proposed in ref. [1] resides in its possible use for a phenomenological description of possible departures from Lorentz symmetry, potentially relevant for certain classes of observations in astrophysics [5], and of a lack of symmetry between particles and antiparticles, which could play a role in describing the observed matter/antimmatter asymmetry.

The theory was analyzed in ref. [1], for the case of two noncommutative fields, on the basis of an analogy with a corresponding $2-d$ noncommutative-harmonic-oscillator problem [2, 3, 6]. This might suggest that standard techniques of analysis would not be applicable, and it also restricts the choice of hamiltonian to one which is compatible with the harmonic-oscillator analogy. We intend to show that the theory can be formulated using the standard techniques based on the Heisenberg equations of motion. For the case of two fields with the hamiltonian adopted in ref. [1] our formulation reproduces the results

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obtained in ref. [1] on the basis of the analogy with the $2-d$ noncommutative-harmonic-oscillator problem. Our formulation however generalizes straightforwardly to the case of an arbitrary number of fields, and, perhaps more importantly, provides us the freedom to consider any type of hamiltonian.

In the next section 2 we briefly comment on the differences between the much studied theory of fields in noncommutative spacetimes and the theory of noncommutative fields proposed in ref. [1]. In section 3, also as a way to introduce some notation, we briefly review the procedure followed in ref. [1], based on the generalization to field theory of the so-called "noncommutative harmonic oscillator construction". In section 4 we present the formulation of the theory based on the Heisenberg equation of motion. In section 5 we show how the construction can be easily generalized to the N-field case. In section 6 we consider other choices of hamiltonian and comment on the fate of Lorentz symmetry in this class of theories.

2. Noncommutative fields vs noncommutative spacetimes

Since it is not uncommon to refer to the rather popular field theories in noncommutative spacetimes as "noncommutative field theories", we find appropriate to start our discussion by clarifying the definition proposed in ref. [1] for a "quantum theory of noncommutative fields", which is basically unrelated to spacetime noncommutativity.

The quantum theory of noncommutative fields [1] assumes in addition to the usual quantum, equal time, commutation relations

$$[\phi(x),\pi(x')]=[\phi^+(x),\pi^+(x')]=i\delta^3(\vec{x}-\vec{x}')\,, \tag{2.1}$$

also non-vanishing commutators between the fields ϕ, ϕ^+, and between the momenta π, π^+:

$$[\phi(x),\phi^+(x')] = \theta\delta^3(\vec{x}-\vec{x}')\,, \tag{2.2}$$

$$[\pi(x),\pi^+(x')] = B\delta^3(\vec{x}-\vec{x}')\,, \tag{2.3}$$

where θ and B are independent constants. As observed in ref. [1], in order to implement both (2.2) and (2.3) with non-vanishing θ and B one should resort to non-hermitian fields.

While in (2.2) and (2.3) it is implicitly assumed that the spacetime coordinates commute (the "noncommutativity of fields" is produced by an appropriate expansion of the fields in creation and annihilation operators), in the better known field theories in noncommutative spacetime one attributes to the coordinates noncommutativity that is in general of the type $[x_\mu,x_\nu]=f_{\mu,\nu}(x)$. A much studied possibility is the one of a "canonical spacetime" (see, e.g., refs. [7]–[10]), with coordinate-independent commutator of the coordinates:

$$[x_\mu,x_\nu]=i\theta_{\mu,\nu}\,. \tag{2.4}$$

There is also growing interest in the "Lie-algebra spacetimes" [8], with $[x_\mu,x_\nu]=\gamma^\alpha_{\mu,\nu}x_\alpha$, and particularly in the κ-Minkowski spacetime [11]–[14, 15]

$$[x_i,x_j] = 0\,, \tag{2.5}$$

$$[x_0,x_i] = i\lambda x_i\,. \tag{2.6}$$

One could ask whether the type of field noncommutativity (2.1)–(2.3) can be somehow (effectively) introduced through spacetime noncommutativity relations of the types (2.4)–(2.6). It easy to verify that this is not the case.[1] In fact, spacetime noncommutativity immediately leads to the fact that all the commutators of fields (including the commutator of a field with itself) are nonvanishing. Consider in particular a general field expansion of the form:

$$\phi_a(x) = \int \frac{d^4p}{(2\pi)^4} \left(\phi_a(p) : e^{ipx} :\right) . \tag{2.7}$$

Choosing, for example, the ordering convention $: e^{ipx} := e^{ip_0x_0} e^{i\vec{p}\vec{x}}$ for exponential factor, in the case of κ-Minkowski spacetime one has for the product of two exponentials:

$$\left(: e^{ipx} :\right)\left(: e^{ikx} :\right) =: e^{i(p\oplus k)x} :, \tag{2.8}$$

$$(p \oplus k) \ = \ \left(p_0 + k_0, \vec{p} + \vec{k}e^{-\lambda p_0}\right) . \tag{2.9}$$

Thus the commutator of two fields assumes the form

$$[\phi_a(x), \phi_b(x)] = \int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} \phi_a(p)\phi_b(k) \Big[e^{i(p_0+k_0)x_0}\left(e^{i(\vec{p}+\vec{k}e^{-\lambda p_0})\cdot\vec{x}} - e^{i(\vec{k}+\vec{p}e^{-\lambda k_0})\cdot\vec{x}}\right)\Big] , \tag{2.10}$$

which does not vanish and actually depends on the field ($[\phi, \phi] = f(\phi)$).

An analogous result is obtained starting from the canonical commutation relations, where one considers

$$\left(: e^{ipx} :\right)\left(: e^{ikx} :\right) = e^{-\frac{i}{2}p^\mu\theta_{\mu,\nu}k^\nu} e^{i(k+p)^\mu x_\mu} , \tag{2.11}$$

so that the commutator of two fields reads

$$[\phi_a(x), \phi_b(x)] = \int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} \phi_a(p)\phi_b(k) \left[(-2i)e^{i(p+k)x} \sin\left(\frac{p^\mu\theta_{\mu,\nu}k^\nu}{2}\right)\right] . \tag{2.12}$$

3. The quantum theory of noncommutative fields from the noncommutative harmonic oscillator

The field theory of noncommutative fields proposed in ref. [1] was based on the hamiltonian

$$H = \int d\vec{x} \left(\pi^+\pi + \vec{\nabla}\phi^+\vec{\nabla}\phi + m^2\phi^+\phi\right) , \tag{3.1}$$

which depends on the fields and on the momenta in a familiar manner.

We note here that (probably as a result of a typographical error) the field denoted by π in ref. [1] is not the momentum conjugate to the field ϕ. Consistency between the form of the hamiltonian and the role of π as conjugate momentum is achieved taking the formulas of ref. [1] and replacing $\pi \to \pi^+$, $\pi^+ \to \pi$. As one sees from the form of the commutator (2.3), this simply leads to a corresponding change of parameter $B \to -B$. At the level of the hamiltonian, the product $\pi^+\pi$ is not invariant under the redefinition

[1] Notice however that the noncommutativity of fields might have nontrivial implications when implemented in the path integral. Here we will not address the path integral formulation of the theory postponing this issue to a future study.

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$\pi \leftrightarrow \pi^+$; however, since this product appears under spatial integration, the resulting effect of the above redefinition is the addition of a constant (-B) to the hamiltonian presented in ref. [1]. This constant does not have a physical role, and can therefore be discarded.

As mentioned, in ref. [1] the theory was analyzed rather indirectly, relying on an analogy with the solution of the quantum-mechanical problem of the noncommutative $2-d$ harmonic oscillator defined by

$$\begin{aligned} H &= \frac{\omega}{2}(q_1^2+q_2^2+p_1^2+p_2^2)\,, && (3.2)\\ [q_1,q_2] &= i\hat{\theta}\,, && (3.3)\\ [p_1,p_2] &= i\hat{B}\,, && (3.4)\\ [q_i,p_j] &= i\delta_{ij}\,, && (3.5)\end{aligned}$$

where $\hat{\theta}=\theta\omega$ and $\hat{B}=B/\omega$.

This noncommutative quantum mechanical system can be transformed into a corresponding commutative system using the maps

$$\begin{aligned} \frac{q_1+iq_2}{\sqrt{2}} &= \eta\epsilon_1 a+\epsilon_2 b^+\,, && (3.6)\\ \frac{p_1+ip_2}{\sqrt{2}} &= -i\epsilon_1 a+i\eta\epsilon_2 b^+\,, && (3.7)\end{aligned}$$

where

$$\begin{aligned} \eta &= \sqrt{1+\left(\frac{\hat{B}-\hat{\theta}}{2}\right)^2}-\left(\frac{\hat{B}-\hat{\theta}}{2}\right), && (3.8)\\ \epsilon_1^2 &= \frac{\hat{B}+\eta}{1+\eta^2}\,, && (3.9)\\ \epsilon_2^2 &= \frac{\eta-\hat{\theta}}{1+\eta^2}\,, && (3.10)\end{aligned}$$

with a, a^+, b, b^+ satisfying the usual canonical commutation rules

$$[a,a^+]=[b,b^+]=1\,, \qquad (3.11)$$

all the other commutators vanishing.

The description of the field ϕ and the momentum field π was obtained in ref. [1] assuming an oscillator of frequency $\omega(p)=\sqrt{\vec{p}^2+m^2}$ for each value of the momentum p:

$$\begin{aligned} \phi &= \int\frac{d\vec{p}}{(2\pi)^3\sqrt{2\omega(p)}}\left[\eta(p)\epsilon_1(p)a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}+\epsilon_2(p)b_{\vec{p}}^+e^{-i\vec{p}\cdot\vec{x}}\right], && (3.12)\\ \pi &= \int\frac{d\vec{p}}{(2\pi)^3}\sqrt{\omega(p)}\left[i\epsilon_1(p)a_{\vec{p}}^+e^{-i\vec{p}\cdot\vec{x}}-i\eta(p)\epsilon_2(p)b_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}\right]. && (3.13)\end{aligned}$$

If $a_{\vec{p}}$ and $b_{\vec{p}}$ satisfy the usual canonical commutation rules:

$$\begin{aligned} [a_{\vec{p}},a_{\vec{p}'}^+] &= (2\pi)^3\delta^3(\vec{p}-\vec{p}')\,, && (3.14)\\ [b_{\vec{p}},b_{\vec{p}'}^+] &= (2\pi)^3\delta^3(\vec{p}-\vec{p}')\,, && (3.15)\end{aligned}$$

then the commutation relations (1-3) are satisfied and the hamiltonian takes the form:

$$H = \int \frac{d\vec{p}}{(2\pi)^3} \left[E_1(\vec{p}) \left(a^+_{\vec{p}} a_{\vec{p}} + \frac{1}{2} \right) + E_2(\vec{p}) \left(b^+_{\vec{p}} b_{\vec{p}} + \frac{1}{2} \right) \right]. \tag{3.16}$$

This expression for the hamiltonian indicates that the theory contains free particles and antiparticles whose energies are respectively

$$E_1(p) = \omega(p) \left[\sqrt{1 + \frac{1}{4} \left(\frac{B}{\omega(p)} - \theta\omega(p) \right)^2} + \frac{1}{2} \left(\frac{B}{\omega(p)} + \theta\omega(p) \right) \right], \tag{3.17}$$

$$E_2(p) = \omega(p) \left[\sqrt{1 + \frac{1}{4} \left(\frac{B}{\omega(p)} - \theta\omega(p) \right)^2} - \frac{1}{2} \left(\frac{B}{\omega(p)} + \theta\omega(p) \right) \right]. \tag{3.18}$$

4. Quantum theory of noncommutative fields from the Heisenberg evolution

In this section we recover the results sketched in the previous section without resorting to the analogy with the noncommutative-harmonic-oscillator problem. In addition to (2.1), (2.2) and (2.3), we will only assume the validity of the Heisenberg equation of motion:

$$\dot{\phi} = -i[\phi, H], \tag{4.1}$$

$$\dot{\pi} = -i[\pi, H], \tag{4.2}$$

that is necessary upon the identification of the hamiltonian with the generator of time evolution.

We observe that by implementing the Heisenberg equation it becomes legitimate to view π as the momentum canonically conjugate to the field ϕ. These quantities in fact can be considered as conjugated only under the equation of motion. We note in particular that the usual conjugation relation

$$\pi = \dot{\phi}^+ \tag{4.3}$$

does not hold here.

From equations (2.1)–(2.3) and (4.1)–(4.2) one has that:

$$\dot{\phi} = \pi^+ + i\theta(\vec{\nabla}^2 - m^2)\phi, \tag{4.4}$$

$$\dot{\pi} = (\vec{\nabla}^2 - m^2)\phi^+ - iB\pi. \tag{4.5}$$

Equations (4.4) and (4.5) clarify the conjugation relations between ϕ and π. Substituting (4.5) in (4.4) one obtains the field equation:

$$\ddot{\phi} - (1 + \theta B)(\vec{\nabla}^2 - m^2)\phi - i\{\theta(\vec{\nabla}^2 - m^2) + B\}\dot{\phi} = 0. \tag{4.6}$$

A solution of the above field equation can be obtained with ϕ in the form

$$\phi(x) = \int \frac{d\vec{p}}{(2\pi)^3} \left[\alpha(p) a_{\vec{p}} e^{-i(\omega_1 t - \vec{p}\cdot\vec{x})} + \beta(p) b^+_{\vec{p}} e^{i(\omega_2 t - \vec{p}\cdot\vec{x})} \right]. \tag{4.7}$$

From equation (4.4) it follows that the conjugate momentum field must be of the form

$$\pi(x) = \qquad (4.8)$$
$$= \int \frac{d\vec{p}}{(2\pi)^3} i \left\{ \alpha(p)[\omega_1 - \theta(\vec{p}^2 + m^2)] a^+_{\vec{p}} e^{i(\omega_1 t - \vec{p}\cdot\vec{x})} - \beta(p) b_{\vec{p}} [\omega_2 + \theta(\vec{p}^2 + m^2)] e^{-i(\omega_2 t - \vec{p}\cdot\vec{x})} \right\}.$$

The request that the field (4.7) is a solution of the field equation implies that

$$\omega_1(p) = \frac{\theta(\vec{p}^2 + m^2) - B}{2} \pm \sqrt{\vec{p}^2 + m^2 + \left[\frac{\theta(\vec{p}^2 + m^2) + B}{2}\right]^2} \qquad (4.9)$$

and that

$$\omega_2(p) = -\frac{\theta(\vec{p}^2 + m^2) - B}{2} \pm \sqrt{\vec{p}^2 + m^2 + \left[\frac{\theta(\vec{p}^2 + m^2) + B}{2}\right]^2}. \qquad (4.10)$$

Moreover the solution of the field equation (4.6) must satisfy the commutation relations (2.1), (2.2) and (2.3), which imply respectively

$$\alpha^2 \left[\omega_1 - \theta(\vec{p}^2 + m^2)\right] + \beta^2 \left[\omega_2 + \theta(\vec{p}^2 + m^2)\right] = 1\,, \qquad (4.11)$$
$$\alpha^2 - \beta^2 = \theta\,, \qquad (4.12)$$
$$\alpha^2 \left[\omega_1 - \theta(\vec{p}^2 + m^2)\right]^2 - \beta^2 \left[\omega_2 + \theta(\vec{p}^2 + m^2)\right]^2 = -B\,. \qquad (4.13)$$

The first two equations of this system can be solved with respect to α and β, obtaining

$$\alpha^2 = \frac{1 + \theta[\theta(\vec{p}^2 + m^2) + \omega_2]}{(\omega_1 + \omega_2)}\,, \qquad (4.14)$$
$$\beta^2 = \frac{1 + \theta[\theta(\vec{p}^2 + m^2) - \omega_1]}{(\omega_1 + \omega_2)}\,. \qquad (4.15)$$

Compatibility with the third equation selects

$$\omega_1(p) = \frac{\theta(\vec{p}^2 + m^2) - B}{2} + \sqrt{\vec{p}^2 + m^2 + \left[\frac{\theta(\vec{p}^2 + m^2) + B}{2}\right]^2}\,, \qquad (4.16)$$
$$\omega_2(p) = -\frac{\theta(\vec{p}^2 + m^2) - B}{2} + \sqrt{\vec{p}^2 + m^2 + \left[\frac{\theta(\vec{p}^2 + m^2) + B}{2}\right]^2}\,. \qquad (4.17)$$

We notice here that equations (4.16)–(4.17) are connected by the transformation $\theta \to -\theta$, $B \to -B$. This transformation maps the "a" particles with energy ω_1 in the "b" particles with energy ω_2, and viceversa. At the level of the fields, as it is also clear from the commutation relations (2.2)–(2.3), the above transformation corresponds to the exchange of the field ϕ with its hermitian conjugated ϕ^+, and viceversa.

It is straightforward now to calculate ϕ and π using the above expressions for α,β,ω_1,ω_2. The formulas we find exactly reproduce the corresponding ones of ref. [1] (briefly discussed in the previous section).

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5. Heisenberg evolution for a theory of noncommutative fields: the N-field case

In this section we extend the procedure described in the previous section to the case in which N real fields are involved. First we must extend the commutation relations (2.1-2.3). This is easily done as follows:

$$[\phi_i(x), \phi_j(x')] = i\Theta_{i,j}\delta^3(\vec{x} - \vec{x}')\,, \tag{5.1}$$

$$[\pi_i(x), \pi_j(x')] = iB_{i,j}\delta^3(\vec{x} - \vec{x}')\,, \tag{5.2}$$

where $i,j = 1,2,\ldots,N$, and $\Theta_{i,j}, B_{i,j}$ are constant antisymmetric matrices. Since we are interested in the free theory, we consider the total hamiltonian to be the sum of the hamiltonian of each field:

$$H = \sum_{i=1}^{N} H_i = \sum_i \int d\vec{x} \left(\frac{1}{2}\pi_i\pi_i + \frac{1}{2}\vec{\nabla}\phi_i\vec{\nabla}\phi_i + \frac{1}{2}m^2\phi_i\phi_i \right). \tag{5.3}$$

From Heisenberg equation, following the same strategy outlined in the previous section we obtain the following system of differential equations:

$$\ddot{\phi}_i(x) - (\delta_{i,k} + B_{i,j}\Theta_{j,k})\left(\vec{\nabla}^2 - m^2\right)\phi_k(x) + \left[\Theta_{i,j}\left(\vec{\nabla}^2 - m^2\right) - B_{i,j}\right]\dot{\phi}_j(x) = 0\,. \tag{5.4}$$

This system of equations allows us to observe that, once the form of the expansions of the fields is taken into account, a standard Lorentz-invariant form of the energy-momentum dispersion relations would require vanishing matrices $\Theta_{i,j}$ and $B_{i,j}$, so that the term going like $\dot{\phi}$ would vanish. But for $\Theta_{i,j} = B_{i,j} = 0$ the fields of course are no longer "noncommutative".

6. Remarks on Lorentz symmetry

For spacetime noncommutativity there has been much interest in the emerging departures from classical Lorentz symmetry, signaled by modified energy/momentum dispersion relations [7, 9, 10, 13, 14, 16]. The presence of departures from classical Lorentz covariance of the "theory of noncommutative fields" here considered, was already pointed out in ref. [1] (see also ref. [17]), indeed through an analysis of the energy/momentum dispersion relations. In our approach the violation of Lorentz symmetry is evident at the level of the field equation (4.6). One could ask if the violation of Lorentz covariance is necessarily implied by the switching-on of the nontrivial $[\phi, \phi^+]$ (and $[\pi, \pi^+]$) commutators. In particular, one might wonder whether an appropriate choice of hamiltonian might compensate for the θ, and B, Lorentz-violating factors in the fields equations coming from the commutation rules.

We consider the rather general class of hamiltonians of the form:

$$\begin{aligned} H = \int d\vec{x} \Big\{ & \alpha_1(\theta,B,m)\pi\pi^+ + \alpha_2(\theta,B,m)\phi\phi^+ + \alpha_3(\theta,B,m)\vec{\nabla}\phi\vec{\nabla}\phi^+ + \alpha_4(\theta,B,m)\times \\ & \times(\pi\phi + \phi^+\pi^+) + \alpha_5(\theta,B,m)(\pi\phi^+ + \phi\pi^+) + \alpha_6(\theta,B,m)\vec{\nabla}\pi\vec{\nabla}\pi^+ + \\ & + \alpha_7(\theta,B,m)(\vec{\nabla}^2\pi\phi + \phi^+\vec{\nabla}^2\pi^+) + \alpha_8(\theta,B,m)(\vec{\nabla}^2\pi^+\phi + \phi^+\vec{\nabla}^2\pi) \Big\}\,, \end{aligned} \tag{6.1}$$

where $\alpha_1, \ldots, \alpha_8$ are real functions of m, θ and B.

The class of hamiltonians (6.1) represents hermitian operators ($H^+ = H$) and includes, as a particular case, the hamiltonian considered in the sections 3 and 4. From the hermitianity of the hamiltonian, and from the fact that the evolution is determined by equations of the type (4.1)–(4.2), it follows that the unitarity of the time evolution is automatically preserved in this framework.

The Heisenberg equations in this case read

$$\dot{\phi} = \mathbf{A}_1\phi + \mathbf{B}_1\phi^+ + \mathbf{C}_1\pi + \mathbf{D}_1\pi^+ \,, \tag{6.2}$$
$$\dot{\pi} = \mathbf{A}_2\phi + \mathbf{B}_2\phi^+ + \mathbf{C}_2\pi + \mathbf{D}_2\pi^+ \,, \tag{6.3}$$

where

$$\mathbf{A}_1 = \alpha_4 - i\alpha_2\theta + (\alpha_7 + i\alpha_3\theta)\vec{\nabla}^2 \,, \tag{6.4}$$
$$\mathbf{B}_1 = \alpha_5 + \alpha_8\vec{\nabla}^2 \,, \tag{6.5}$$
$$\mathbf{C}_1 = -i\theta\alpha_5 - i\theta\alpha_8\vec{\nabla}^2 \,, \tag{6.6}$$
$$\mathbf{D}_1 = \alpha_1 - i\alpha_4\theta - (\alpha_6 + i\alpha_7\theta)\vec{\nabla}^2 \,, \tag{6.7}$$
$$\mathbf{A}_2 = -iB\alpha_5 - iB\alpha_8\vec{\nabla}^2 \,, \tag{6.8}$$
$$\mathbf{B}_2 = -\alpha_2 - i\alpha_4 B + (\alpha_3 - i\alpha_7 B)\vec{\nabla}^2 \,, \tag{6.9}$$
$$\mathbf{C}_2 = -\alpha_4 - i\alpha_1 B - (\alpha_7 - i\alpha_6 B)\vec{\nabla}^2 \,, \tag{6.10}$$
$$\mathbf{D}_2 = -\alpha_5 - \alpha_8\vec{\nabla}^2 \,. \tag{6.11}$$

We start by analyzing the simple case of $\alpha_5 = \alpha_8 = 0$. In this case from (6.2)–(6.3) one obtains the field equations:

$$\ddot{\phi} = [\mathbf{D}_1\mathbf{B}_2^+ - \mathbf{A}_1\mathbf{C}_2^+]\phi + [\mathbf{A}_1 + \mathbf{C}_2^+]\dot{\phi} \,, \tag{6.12}$$
$$\ddot{\pi} = [\mathbf{D}_1\mathbf{B}_2^+ - \mathbf{A}_1\mathbf{C}_2^+]\pi + [\mathbf{A}_1 + \mathbf{C}_2^+]\dot{\pi} \,. \tag{6.13}$$

A necessary condition for the covariance of the above equations is that

$$\mathbf{A}_1 = -\mathbf{C}_2^+ \,, \tag{6.14}$$

which implies the relations

$$\alpha_2\theta = \alpha_1 B \tag{6.15}$$
$$\alpha_3\theta = \alpha_6 B \,. \tag{6.16}$$

These relations, for nonzero θ and B, are not compatible with the proper $\theta, B \to 0$ limit ($\alpha_1 = \alpha_3 = 1$, $\alpha_2 = m^2$, $\alpha_4 = \alpha_6 = \alpha_7 = 0$).

In the case $\alpha_5, \alpha_8 \neq 0$ the field equations assume a much more involved form than that of (6.12) and (6.13), but again it can be shown that, for nonzero θ and B, the field equations are incompatible with Lorentz covariance. In order to achieve less severe departures from Lorentz symmetry, while still working within the framework of Heisenberg evolution, one might have to adopt field commutation relations different from the ones here considered (see e.g. ref. [18]).

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